Integrals

1. (i) $\int x^n dx = \frac{x^{n+1}}{(n+1)} + C, n \neq -1$ (ii) $\int \frac{1}{x} dx = \log|x| + C, x \neq 0$

(iii) $\int e^x dx = e^x + C$ (iv) $\int a^x dx = \frac{a^x}{\log a} + C$

2. (i) $\int \sin x\, dx = -\cos x + C$ (ii) $\int \cos x\, dx = \sin x + C$

(iii) $\int \tan x\, dx = \log|\sec x| + C$ (iv) $\int \cot x dx = \log|\sin x| + C$

(v) $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log|\tan(\pi/4 + x/2)| + C$

(vi) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C = \log|\tan(x/2)| + C$

(vii) $\int \sec^2 x dx = \tan x + C$

(viii) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(ix) $\int \sec x \tan x dx = \sec x + C$

(x) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$

3. (i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ (ii) $-\int \frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$

(iii) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ (iv) $-\int \frac{1}{1+x^2} dx = \cot^{-1} x + C$

(v) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(vi) $-\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + C$

4. $\int (ax+b)^n\, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{(n+1)} + C$

5. **Partial fractions of rational functions** Rational functions are of the form $\frac{P(x)}{Q(x)}$, Where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If degree of $Q(x)$ is less than or equal to the degree of

$P(x)$, then divide $P(x)$ by $Q(x)$ by using long division method till the degree of new $P(x)$ (*i.e.*, quotient) is less than the degree of $Q(x)$. Now, use the following cases :

(i) If $Q(x)$ is the product of linear factors, then $\frac{P(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$

(ii) If $Q(x)$ is the product of repeated linear factors, then $\frac{P(x)}{(x+a)(x+b)^2} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$

(iii) If $Q(x)$ is the product of linear and quadratic factors, then $\frac{P(x)}{(x+a)(px^2+qx+r)} = \frac{A}{x+a} + \frac{Bx+C}{px^2+qx+r}$

(iv) If $Q(x)$ is the product of quadratic factors, then $\frac{P(x)}{(ax^2+bx+c)(px^2+qx+r)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{px^2+qx+r}$

6. **Integration by parts** (Integral of the product of two functions)

$\int (uv)\, dx = u\int v dx - \int \left[\frac{d}{dx}(u).\int v dx\right] dx =$ (First function $\times$ Integration of IInd function)

$-\int$ [(Differential coefficient of Ist function)

$\times$ (Integration of IInd function)]dx

Note (*i*) *We can also choose the first function as the function which comes first in the word ILATE, where*

I = Inverse trigonometric function,

L = Logarithmic function

A = Algebraic function

T = Trigonometric function,

E = Exponential function

(ii) *If both the functions are trigonometrical, then take that function as v whose integral is simpler.*

(iii) *If both the functions are algebraic then take that function as u whose differentiation is simpler.*

7. (i) $\int \frac{dx}{(x^2-a^2)} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ (ii) $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iii) $\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

8. (i) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$ (ii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(iii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

9. (i) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(ii) $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iii) $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

10. (i) Integrals of the form $\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$

Method Put the denominator in the form $a\,[(x + \alpha)^2 + \beta^2]$ and then integrate.

(ii) Integrals of the form $\int \frac{dx}{(ax^2 + bx + c)}$

Method Put $(ax^2 + bx + c)$ in the form $a[(x + \alpha)^2 + \beta^2]$ and then integrate.

11. (i) Integrals of the form $\int (px + q)\sqrt{(ax^2 + bx + c)}dx$

Method Put $(px + q) = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B.$

Comparing the coefficients of like powers of x on both sides and find A and B. The integrands are now easily integrable.

(ii) Integrals of the form $\int \frac{(px^2 + qx + r)}{(ax^2 + bx + c)}dx$

Method Put $(px^2 + qx + r) = A\,(ax^2 + bx + c) + B \cdot \frac{d}{dx}(ax^2 + bx + c) + C$

Comparing the coefficients of like powers of x on both sides and find A, B and C. The integrands are now easily integrable.

12. Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, where P and Q are linear or quadratic expressions in x.

Method (i) When Q is linear, put $Q = t^2$

(ii) When Q is quadratic and P is linear, put $P = \frac{1}{t}$.

(iii) When Q and P are both purely quadratic, put $x = \frac{1}{t}$.

13. Integrals of the form $\int \frac{dx}{(a\sin^2 x + b\cos^2 x)}$.

Method Divide numerator and denominator by $\cos^2 x$, put $\tan x = t$ and integrate.

14. Integrals of the form $\int \frac{dx}{(a + b\cos x)}$ or $\int \frac{dx}{(a + b\sin x)}$

Method Put $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$ and $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$

Now, put $\tan(x/2) = t$.

15. Integrals of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$

Method Let numerator $= A$ (denominator) + B. (Derivative of denominator) Equating the coefficients of $\sin x$ and $\cos x$ on both sides, we get A and B and then integrate.

Definite Integrals

1. Integrals as Limit of the Sum

$$\int_a^b f(x)dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a + (n-1)h\}],$$

where, $nh = (b - a)$

2. Some Useful Results

(i) $1 + 2 + 3 + \ldots + (n-1) = \frac{(n-1)n}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \frac{(n-1)^2 n^2}{4}$

(iv) $\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin\{a + (n-1)h\}$
$= \frac{\sin(nh/2)}{\sin(h/2)} \sin\left\{a + \frac{1}{2}(n-1)h\right\}$

(v) $\cos a + \cos(a+h) + \cos(a+2h) + \ldots + \cos\{a + (n-1)h\}$
$= \frac{\sin(nh/2)}{\sin(h/2)} \cos\left\{a + \frac{1}{2}(n-1)h\right\}$.

3. Definite Integrals

(i) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a <<< b$

(iii) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(iv) $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

(v) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

(vi) $\int_0^{2a} f(x)dx = \begin{cases} 2\cdot\int_0^a f(x)dx, & \text{when } f(2a - x) = f(x) \\ 0, & \text{when } f(2a - x) = -f(x) \end{cases}$

(vii) $\int_{-a}^a f(x)dx = \begin{cases} 2\cdot\int_0^a f(x)dx, & \text{when } f(x) \text{ is even} \\ 0, & \text{when } f(x) \text{ is odd} \end{cases}$